

An Analytical Study on the Multi-critical Behaviour and Related Bifurcation Phenomena for Relativistic Black Hole Accretion

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Abstract We apply the theory of algebraic polynomials to analytically study the transonic properties of general relativistic hydrodynamic axisymmetric accretion onto non-rotating astrophysical black holes. For such accretion phenomena, the conserved specific energy of the flow, which turns out to be one of the two first integrals of motion in the system studied, can be expressed as a 8^{th} degree polynomial of the critical point of the flow configuration. We then construct the corresponding Sturm's chain algorithm to calculate the number of real roots lying within the astrophysically relevant domain of \mathbb{R} . This allows, for the first time in literature, to *analytically* find out the maximum number of physically acceptable solution an accretion flow with certain geometric configuration, space-time metric, and equation of state can have, and thus to investigate its multi-critical properties *completely analytically*, for accretion flow in which the location of the critical points can not be computed without taking recourse to the numerical scheme. This work can further be generalized to analytically calculate the maximal number of equilibrium points certain autonomous dynamical system can have in general. We also demonstrate how the transition from a mono-critical to multi-critical (or vice versa) flow configuration can be realized through

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the saddle-centre bifurcation phenomena using certain techniques of the catastrophe theory.

Keywords accretion, accretion discs · black hole physics · hydrodynamics · gravitation

1 Introduction

In order to satisfy the inner boundary conditions imposed by the event horizon, accretion onto astrophysical black holes exhibit transonic properties in general [1]. A physical transonic accretion solution can mathematically be realized as critical solution on the phase portrait [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Multi-critical accretion may be referred to the specific category of accretion flow configuration having multiple critical points accessible to the accretion solution. For certain astrophysically relevant values of the initial boundary conditions, low angular momentum sub-Keplerian axisymmetric black hole accretion can have at most three critical points all together – where two saddle type critical points accommodate one centre type critical point in between them [1, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. Transonic solution passing through the aforementioned two critical points can be joined through a stationary shock generated as a consequence of the presence of the angular momentum barrier [18, 35, 29, 36, 32, 33]. The existence of such weakly rotating accretion in realistic astrophysical environment have also been observed [37, 38, 39, 40, 41, 42]. A complete investigation of the multi-critical shocked accretion flow around astrophysical black holes necessitates the numerical integration of the nonlinear stationary equations describing the velocity phase space behaviour of the flow.

However, for all the importance of transonic flows, there exists as yet no general mathematical prescription allowing one a direct analytical understanding of the nature of the multi-criticality without having to take recourse to the existing semi-analytic approach of numerically finding out the total number of physically acceptable critical points the accretion flow can have.

This is precisely the main achievement of our work presented in this paper. Using the theory of algebraic polynomials, we developed a mathematical algorithm capable of finding the number of physically acceptable solution a polynomial can have, for any arbitrary large value of n (n being the degree of the polynomial). For a specified set of values of the initial boundary conditions, we mathematically predict whether the flow will be multi-critical (more than one real physical roots for the polynomial) or not. This paper, thus, purports to address that particular issue of investigating the transonicity of a general relativistic flow structure around non rotating black holes without encountering the usual semi-analytic numerical techniques, and to derive some predictive insights about the qualitative character of the flow, and in relation to that, certain physical features of the multi-criticality of the flow will also be addressed. In our work, we would like to develop a complete analytical formalism to investigate the critical behaviour of the general relativistic low angular momentum inviscid axisymmetric advective hydrodynamic accretion flow around a non rotating black hole.

To accomplish the aforementioned task, we first construct the equation describing the space gradient of the dynamical flow velocity of accreting matter. Such equation is isomorphic to a first order autonomous dynamical system. Application of the fixed point analysis enables to construct an 8th degree algebraic equation for the space variable along which the flow streamlines are defined to possess certain first integrals of motion. The constant coefficients for each term in that equation are functions of astrophysically relevant initial boundary conditions. Such initial boundary conditions span over a certain domain on the real line \mathbb{R} – effectively, as individual sub-domain of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ for the polytropic accretion. The solution of aforesaid equation would then provide the critical (and consequently, the sonic) point r_c . The critical points itself are permissible only within a certain open interval $]r_g, L_{\rightarrow\infty}[$, where r_g is the radius of the event horizon and $L_{\rightarrow\infty}$ is the physically acceptable maximally allowed limit on the value of a critical point.

Since for polynomials of degree $n > 4$, analytical solutions are not available, we use the Sturm's theorem (a corollary of the Sylvester's theorem), to construct the Sturm's chain algorithm, which can be used to calculate the number of real roots (lying within a certain sub-domain of \mathbb{R}) for a polynomial of any countably finite arbitrarily large integral n , subjected to certain sub-domains of constant co-efficients. The problem now reduces to identify the polynomials in r_c with the Sturm's sequence, and to find out the maximum number of physically acceptable solution an accretion flow with certain geometric configuration, space-time metric, and equation of state can have, and thus to investigate its multi-critical properties completely analytically, for accretion flow in which the critical points can not be computed analytically. Our work, as we believe, has significant importance, because for the first time in the literature, we provide a purely analytical method, by applying certain theorem of algebraic polynomials to check whether certain astrophysical hydrodynamic accretion may undergo more than one sonic transitions.

We further demonstrate how the transition of number of critical points may be taken into account considering the bifurcation phenomenon in the parameter space. The transition of number of critical points in this case is associated with the merging and destruction (or emergence and separating apart, viewing in the other way round) of a saddle-centre pair, i.e. a saddle-centre bifurcation common in conservative systems, which may be tracked down using technique of catastrophe theory. The bifurcation lines in the parameter space exactly conform with the transition boundaries of the across which the number of critical points changes.

2 First Integral of Motion as a Polynomial in Critical Radius

Following standard literature, we assume that the axisymmetric accretion flow has a radius dependent local thickness $H(r)$, and its central plane coincides with the equatorial plane of the black hole. It is common practice in accretion disc theory ([43, 44, 45, 46, 26, 47, 48, 49, 50, 51]) to use the vertically integrated model in describing the black hole accretion discs where the equations of motion apply to the equatorial plane of the black hole assuming the flow to be in hydrostatic equilibrium along transverse direction. We follow the same procedure here. The thermodynamic flow variables are

averaged over the disc height, i.e. a thermodynamic quantity y used in our model is vertically integrated over the disc height and averaged as $\bar{y} = \int_0^{H(r)} y dh / \int_0^{H(r)} dh$.

We follow [52] to derive an expression for the disc height $H(r)$ in our geometry since the relevant equations in [52] are non-singular on the horizon and can accommodate both axial and quasi spherical flow geometry. The disc height comes out to be [32],

$$H(r) = \frac{c_s r}{\lambda} \sqrt{\frac{2(\gamma-1)(1-u^2)[r^3 - \lambda^2(r-2)]}{\gamma[\gamma - (1+c_s^2)](r-2)}} \quad (1)$$

where λ and γ are the specific flow angular momentum and the adiabatic index of the flow, respectively. u and c_s being the dynamical flow velocity and the speed of propagation of the acoustic perturbation (adiabatic sound speed) embedded within the accretion flow. In this work, we employ polytropic accretion. However, polytropic accretion is not the only choice to describe the general relativistic axisymmetric black-hole accretion. Equations of state other than the adiabatic one, such as the isothermal equation [27] or two temperature plasma [53] have also been used to study the black-hole accretion flow.

For accretion flow of aforementioned category, two first integrals of motion along the streamline, viz, the dimensionless conserved specific flow energy i.e., the energy per unit mass which actually is scaled by the rest mass of the flow \mathcal{E} , and the mass accretion rate \dot{M} , may be obtained as (the radial distance r here is actually scaled by the factor GM_{BH}/c^2 , and all the velocities, both u as well as c_s have been scaled by the velocity of light c in vacuum. M_{BH} is the mass of the black hole. Natural geometric unit has been used where the values of all fundamental constants have been taken to be unity, see, e.g., [32] for further detail)

$$\mathcal{E} = \left[\frac{(\gamma-1)}{\gamma - (1+c_s^2)} \right] r \sqrt{\frac{r-2}{r^3 - \lambda^2(r-2)}} \frac{1}{\sqrt{1-u^2}}, \quad (2)$$

$$\dot{M} = \frac{4\pi\rho c_s r^{\frac{3}{2}} u}{\lambda} \sqrt{\frac{2(\gamma-1)[r^3 - \lambda^2(r-2)]}{\gamma[\gamma - (1+c_s^2)]}}, \quad (3)$$

where ρ is the mass density. The expression for \mathcal{E} is obtained by integrating the stationary part of the Euler equation and the expression for \dot{M} is obtained by integrating the stationary part of the continuity equation (by properly taking care of the flow thickness). The conserved specific entropy accretion rate $\dot{\mathcal{M}}$ is computed as a quasi constant multiple of \dot{M} as:

$$\dot{\mathcal{M}} = 4\pi \left(\frac{1}{\lambda} \sqrt{\frac{2}{\gamma}} \right) \left[\frac{c_s}{\left(1 - \frac{c_s^2}{\gamma-1}\right)^{\frac{1}{2}}} \right]^{\frac{\gamma+1}{\gamma-1}} u r [r^4 - \lambda^2 r(r-2)]^{\frac{1}{2}}, \quad (4)$$

We thus have two primary first integrals of motion along the streamline – the specific energy of the flow \mathcal{E} and the mass accretion rate \dot{M} . Even in the absence of creation or annihilation of matter, the entropy accretion rate $\dot{\mathcal{M}}$ is not a generic first integral of motion. As the expression for $\dot{\mathcal{M}}$ contains the quantity $K \equiv p/\rho^\gamma$ (p being the flow

pressure), which is a measure of the specific entropy of the flow, the entropy accretion rate $\dot{\mathcal{M}}$ remains constant throughout the flow only if the entropy per particle remains locally invariant. This condition may be violated if the accretion is accompanied by a shock. Thus $\dot{\mathcal{M}}$ is conserved for shock free polytropic accretion and becomes discontinuous (actually, increases) at the shock location, if such a shock is formed.

The gradient of the acoustic velocity c_s as well as the dynamical velocity u can be obtained by differentiating the expression for the entropy accretion rate and the mass accretion rate respectively:

$$\frac{dc_s}{dr} = -\frac{c_s(\gamma-1)}{(\gamma+1)} \left[\frac{\gamma - (1+c_s^2)}{u} \frac{du}{dr} + f_1(r, \lambda) \right], \quad (5)$$

where

$$f_1(r, \lambda) = \frac{3r^3 - 2\lambda^2 r + 3\lambda^2}{r^4 - \lambda^2 r(r-2)}. \quad (6)$$

$$\frac{du}{dr} = \frac{(\frac{2}{\gamma+1})c_s^2 f_1(r, \lambda) - f_2(r, \lambda)}{\frac{u}{1-u^2} - \frac{2c_s^2}{u(\gamma+1)}} = \frac{\mathcal{N}(r, \lambda, c_s)}{\mathcal{D}(u, c_s)}, \quad (7)$$

where

$$f_2(r, \lambda) = \frac{2r-3}{r(r-2)} - \frac{2r^3 - \lambda^2 r + \lambda^2}{r^4 - \lambda^2 r(r-2)}. \quad (8)$$

A real physical transonic flow must be smooth everywhere, except possibly at a shock. Hence, if the denominator $\mathcal{D}(u, c_s)$ of Eq. (7) vanishes at a point, the numerator $\mathcal{N}(r, \lambda, c_s)$ must also vanish at that point to ensure the physical continuity of the flow. One therefore arrives at the *critical point* conditions by making $\mathcal{D}(u, c_s)$ and $\mathcal{N}(r, \lambda, c_s)$ of Eq. (7) simultaneously equal to zero. We thus obtain the critical point conditions as

$$u_c = \pm \sqrt{\frac{f_2(r_c, \lambda)}{f_1(r_c, \lambda) + f_2(r_c, \lambda)}}; \quad c_c = \pm \sqrt{\frac{\gamma+1}{2} \left[\frac{f_2(r_c, \lambda)}{f_1(r_c, \lambda)} \right]}; \quad (9)$$

where $u_c \equiv u(r_c)$ and $c_c \equiv c_s(r_c)$, r_c being the location of the critical point. $f_1(r_c, \lambda)$ and $f_2(r_c, \lambda)$ are defined as:

$$f_1(r_c, \lambda) = \frac{3r_c^3 - 2\lambda^2 r_c + 3\lambda^2}{r_c^4 - \lambda^2 r_c(r_c - 2)}, \quad f_2(r_c, \lambda) = \frac{2r_c - 3}{r_c(r_c - 2)} - \frac{2r_c^3 - \lambda^2 r_c + \lambda^2}{r_c^4 - \lambda^2 r_c(r_c - 2)} \quad (10)$$

Clearly, the critical points are not coincident with the sonic points since $M_c = (u_c/c_c) < 1$. This is a consequence of the choice of the equation of state. The adiabatic equation of state used in this work produces non constant (with respect to the radial space direction) sound speed. Since the disc height contains the sound speed and the thermodynamic quantities calculated in the accretion flow have been averaged over the flow thickness, non constant sound speed accounts for the non-isomorphism of the critical points and the sonic points. If one uses the sound speed obtained from isothermal equation of state, or a flow geometry different from the configuration in the vertical equilibrium as has been assumed here, the critical points will coincide with the sonic points, see, e. g., [31, 34] for further detail.

We substitute the explicit value of u_c and c_c from Eq. (9) to the expression for the specific energy \mathcal{E} in Eq. (2) to derive the explicit form of the energy first integral polynomial in r_c as:

$$\begin{aligned}
& r_c^8 \{-36(-1 + \mathcal{E}^2)(-1 + \gamma)^2\} + r_c^7 \{12(-1 + \gamma)(-17(-1 + \gamma) + \mathcal{E}^2(-11 + 13\gamma))\} + \\
& r_c^6 \{-24(-1 + \gamma)^2(-16 + \lambda^2) + \mathcal{E}^2(-121 + 60\lambda^2 + \gamma(286 - 96\lambda^2) + \gamma^2(-169 + 36\lambda^2))\} \\
& + r_c^5 \{-2(120 + (-86 + 163\mathcal{E}^2)\lambda^2 + \gamma^2(120 + (-86 + 99\mathcal{E}^2)\lambda^2) - 2\gamma(120 + (-86 + 133\mathcal{E}^2)\lambda^2))\} \\
& + r_c^4 \{\lambda^2(-460(-1 + \gamma)^2 + \mathcal{E}^2(588 - 25\lambda^2 + \gamma^2(356 - 9\lambda^2) + \gamma(-976 + 30\lambda^2)))\} \\
& + r_c^3 \{4\lambda^2(136(-1 + \gamma)^2 + \mathcal{E}^2(-88 + 45\lambda^2 + \gamma(148 - 52\lambda^2) + \gamma^2(-52 + 15\lambda^2)))\} \\
& + r_c^2 \{-4\lambda^2(60 + 121\mathcal{E}^2\lambda^2 + \gamma^2(60 + 37\mathcal{E}^2\lambda^2) - 2\gamma(60 + 67\mathcal{E}^2\lambda^2))\} \\
& + r_c \{32\mathcal{E}^2(18 - 19\gamma + 5\gamma^2)\lambda^4\} + \{-64\mathcal{E}^2(-2 + \gamma)^2\lambda^4\} = 0
\end{aligned} \tag{11}$$

The above equation, being an $n = 8$ polynomial, is non analytically solvable. Being equipped with the details of the Sturm theorem and its appropriate application in the next section (§3), in §4 we will demonstrate how we can analytically find out the number of physically admissible real roots for this polynomial, and can investigate the transonicity of the flow.

3 Sturm theorem and generalized sturm sequence (chain)

In this section we will elaborate the idea of the generalized Sturm sequence/chain, and will discuss its application in finding the number of roots of a algebraic polynomial equations with real co-efficients. Since the central concept of this theorem is heavily based on the idea of the greatest common divisor of a polynomial and related Euclidean algorithm, we start our discussion by clarifying such concept in somewhat great detail for the convenience of the reader.

3.1 Greatest common divisor for two numbers

Given two non-zero integers z_1 and z_2 , one defines that z_1 divides z_2 , if and only if there exists some integer $z_3 \in \mathbb{Z}$ such that:

$$z_2 = z_3 z_1 \tag{12}$$

The standard notation for the divisibility is as follows:

$$z_1 | z_2 \text{ means ' } z_1 \text{ divides } z_2 \text{ ' } \tag{13}$$

The concept of divisibility applies to the polynomials as well, we treat such situations in the subsequent paragraphs.

Now consider two given integers z_1 and z_2 , with at least one of them being a non-zero number. The 'greatest common divisor' (or the 'greatest common factor' or the

‘highest common factor’) of z_1 and z_2 , denoted by $gcd(z_1, z_2)$, is the positive integer $z_d \in \mathbb{Z}$, which satisfies:

$$\begin{aligned} & \text{i) } z_d | z_1 \text{ and } z_d | z_2. \\ & \text{ii) For any other } z_c \in \mathbb{Z}, \text{ if } z_c | z_1 \text{ and } z_c | z_2 \\ & \quad \text{then } z_c | z_d \end{aligned} \quad (14)$$

In other words, the greatest common divisor $gcd(z_1, z_2)$ of two non zero integers z_1 and z_2 is the largest possible integer that divides both the integers without leaving any remainder. Two numbers z_1 and z_2 are called ‘co-prime’ (alternatively, ‘relatively prime’), if:

$$gcd(z_1, z_2) = 1 \quad (15)$$

The idea of a greatest common divisor can be generalized by defining the greater common divisor of a non empty set of integers. If $\mathcal{S}_{\mathbb{Z}}$ is a non-empty set of integers, then the greatest common divisor of $\mathcal{S}_{\mathbb{Z}}$ is a positive integer z_d such that:

$$\begin{aligned} & \text{i) If } z_d | z_1 \text{ for all } z_1 \in \mathcal{S}_{\mathbb{Z}} \\ & \text{ii) If } z_2 | z_1, \text{ for all } z_1 \in \mathcal{S}_{\mathbb{Z}}, \text{ then } z_2 | z_d \end{aligned} \quad (16)$$

then we denote $z_d = gcd(\mathcal{S}_{\mathbb{Z}})$.

3.2 Euclidean algorithm

Euclidean algorithm (first described in detail in Euclid’s ‘Elements’ in 300 BC, and is still in use, making it the oldest available numerical algorithm still in common use) provides an efficient procedure for computing the greatest common divisor of two integers. Following Stark [54], below we provide a simplified illustration of the Euclidean algorithm for two integers:

Let us first set a ‘counter’ i for counting the steps of the algorithm, with initial step corresponding to $i = 0$. Let any i th step of the algorithm begins with two non-negative remainders r_{i-1} and r_{i-2} with the requirement that $r_{i-1} < r_{i-2}$, owing to the fact that the fundamental aim of the algorithm is to reduce the remainder in successive steps, to finally bring it down to the zero in the ultimate step which terminates the algorithm. Hence, for the dummy index i , at the first step we have:

$$r_{-2} = z_2 \text{ and } r_{-1} = z_1 \quad (17)$$

the integers for which the greatest common divisor is sought for. After we divide z_2 by z_1 (operation corresponds to $i = 1$), since z_2 is not divisible by z_1 , one obtains:

$$r_{-2} = q_0 r_{-1} + r_0 \quad (18)$$

where r_0 is the remainder and q_0 be the quotient.

For any arbitrary i th step of the algorithm, the aim is to find a quotient q_j and remainder r_i , such that:

$$r_{i-2} = q_i r_{i-1} + r_i, \text{ where } r_i < r_{i-1} \quad (19)$$

at some step $i = j$ (common sense dictates that j can not be infinitely large), the algorithm terminates because the remainder becomes zero. Hence the final *non-zero* remainder r_{j-1} will be the greatest common divisor of the corresponding integers.

We will now illustrate the Euclidean algorithm for finding the greatest common divisor for two polynomials.

3.3 Greatest common divisor and related Euclidean algorithm for polynomials

Let us first define a polynomial to be ‘monic’ if the co-efficient of the term for the highest degree variable in the polynomial is unity (one). Let us now consider $p_1(x)$ and $p_2(x)$ to be two nonzero polynomials with co-efficient from a field \mathbb{F} (field of real, complex, or rational numbers, for example). A greatest common divisor of $p_1(x)$ and $p_2(x)$ is defined to be the monic polynomial $p_d(x)$ of highest degree such that $p_d(x)$ divides both $p_1(x)$ and $p_2(x)$. It is obvious that \mathbb{F} be field and $p_d(x)$ be a monic, are necessary hypothesis.

In more compact form, a greatest common divisor of two polynomials $p_1, p_2 \in \mathbb{R}[\mathbb{X}]$ is a polynomial $p_d \in \mathbb{R}[\mathbb{X}]$ of greatest possible degree which divides both p_1 and p_2 . Clearly, p_d is not unique, and is only defined upto multiplication by a non zero scalar, since for a non zero scalar $c \in \mathbb{R}$, if p_d is a $gcd(p_1, p_2 \in \mathbb{R}[\mathbb{X}])$, so as cp_d . Given polynomials $p_1, p_2 \in \mathbb{R}[\mathbb{X}]$, the division algorithm provides polynomials $p_3, p_4 \in \mathbb{R}[\mathbb{X}]$, with $deg(p_4) < deg(p_3)$ such that

$$p_1 = p_3 p_2 + p_4 \quad (20)$$

Then, if p_d is $gcd(p_1, p_2)$, if and only if p_d is $gcd(p_2, p_4)$ as is obvious.

One can compute the gcd of two polynomials by collecting the common factors by factorizing the polynomials. However, this technique, although intuitively simple, almost always create a serious practical threat while making attempt to factorize the large high degree polynomials in reality. Euclidean algorithm appears to be relatively less complicated and a faster method for all practical purposes. Just like the integers as shown in the previous subsection, Euclid’s algorithm can directly be applied for the polynomials as well, with decreasing degree for the polynomials at each step. The last non-zero remainder, after made monic if necessary, comes out to be the greatest common divisor of the two polynomials under consideration.

Being equipped with the concept of the divisibility, gcd and the Euclidean algorithm, we are now in a position to define the Sturm theorem and to discuss its applications.

3.4 The Sturm Theorem: The purpose and the definition

The Sturm theorem is due to Jacques Charles Francois Sturm, a Geneva born French mathematician and a close collaborator of Joseph Liouville. The Sturm theorem, published in 1829 in the eleventh volume of the ‘Bulletin des Sciences de Ferussac’ under the title ‘Memoire sur la resolution des equations numeriques’¹. The Sturm theorem,

¹ According to some historian, the theorem was originally discovered by Jean Baptist Fourier, well before Sturm, on the eve of the French revolution.

which is actually a root counting theorem, is used to find the number of real roots over a certain interval of a algebraic polynomial with real co-efficient. It can be stated as:

Theorem 1 *The number of real roots of an algebraic polynomial with real coefficient whose roots are simple over an interval, the endpoints of which are not roots, is equal to the difference between the number of sign changes of the Sturm chains formed for the interval ends.*

Hence, given a polynomial $p \in \mathbb{R}[\mathbb{X}]$, if we need to find the number of roots it can have in a certain open interval $]a, b[$, a and b not being the roots of f , we then construct a sequence, called ‘Sturm chain’, of polynomials, called the generalized strum chains. Such a sequence is derived from p using the Euclidean algorithm. For the polynomial p as described above, the Sturm chain $p_0, p_1 \dots$ can be defined as:

$$\begin{aligned} p_0 &= p \\ p_1 &= p' \\ p_n &= -\text{rem}(p_{n-2}, p_{n-1}), n \geq 2 \end{aligned} \quad (21)$$

where $\text{rem}(p_{n-2}, p_{n-1})$ is the remainder of the polynomial p_{n-2} upon division by the polynomial p_{n-1} . The sequence terminates once one of the p_i becomes zero. We then evaluate this chain of polynomials at the end points a and b of the open interval. The number of roots of p in $]a, b[$ is the difference between the number of sign changes on the chain of polynomials at the end point a and the number of sign changes at the end point b . Thus, for any number t , if $N_{p(t)}$ denotes the number of sign changes in the Sturm chain $p_0(t), p_1(t), \dots$, then for real numbers a and b that (both) are not roots of p , the number of distinct real roots of p in the open interval $]a, b[$ is $[N_{p(a)} - N_{p(b)}]$. By making $a \rightarrow -\infty$ and $b \rightarrow +\infty$, one can find the total number of roots p can have on the entire domain of \mathbb{R} .

A more formal definition of the Sturm theorem, as a corollary of the Sylvester’s theorem, is what follows:

Definition Let R be the real closed field, and let p and P be in $R[X]$. The Sturm sequence of p and P is the sequence of polynomials (p_0, p_1, \dots, p_k) defined as follows:

$$\begin{aligned} p_0 &= p, p_1 = p'P \\ p_i &= p_{i-1}q_i - p_{i-2} \text{ with } q_i \in R[X] \text{ and } \deg(p_i) < \deg(p_{i-1}) \text{ for } i = 2, 3, \dots, k, \\ p_k &\text{ is a greatest common divisor of } p \text{ and } p'P. \end{aligned}$$

Given a sequence (a_0, \dots, a_k) of elements of R with $a_0 \neq 0$, we define the number of sign changes in the sequence (a_0, \dots, a_k) as follows: count one sign change if $a_i a_l < 0$ with $l = i + 1$ or $l > i + 1$ and $a_j = 0$ for every $j, i < j < l$.

If $a \in R$ is not a root of p and (p_0, \dots, p_k) is the Sturm sequence of p and P , we define $v(p, P; a)$ to be the number of sign changes in $(p_0(a), \dots, p_k(a))$.

Theorem 2 (Sylvester’s Theorem²) *Let R be a real closed field and let p and P be two polynomials in $R[X]$. Let $a, b \in R$ be such that $a < b$ and neither a nor b are roots of p . Then the difference between the number of roots of p in the interval $]a, b[$ for which P is positive and the number of roots of p in the interval $]a, b[$ for which P is negative, is equal to $v(p, P; a) - v(p, P; b)$*

² As stated in [55].

Corollary 1 (Sturm's Theorem): *Let R be a real closed field and $p \in R[X]$. Let $a, b \in R$ be such that $a < b$ and neither a nor b are roots of p . Then the number of roots of p in the interval $]a, b[$ is equal to $v(p, 1; a) - v(p, 1; b)$.*

The proof of these two theorems are given in the Appendix I.

4 Number of available critical points for relativistic accretion

We first write down the complete expression for the Sturm chains. Then for a suitable parameter set $[\mathcal{E}, \lambda, \gamma]$, we can find the difference of the sign change of the Sturm chains at the open interval left boundary, i.e., at the event horizon and at the right boundary, i.e., at some suitably chosen large distance, say, 10^8 gravitational radius (which is such a large distance that beyond which practically no critical point is expected to form unless the specific flow energy has an extremely low value, i.e., very cold accretion flow), to find the number of critical points the accretion flow can have.

The form of the original polynomial has already been explicitly expressed using left hand side of Eq. 11. We now construct the Sturm chains as:

$$\begin{aligned}
 p_0(r) &= a_8 r^8 + a_7 r^7 + a_6 r^6 + a_5 r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 \\
 p_1(r) &= 8a_8 r^7 + 7a_7 r^6 + 6a_6 r^5 + 5a_5 r^4 + 4a_4 r^3 + 3a_3 r^2 + 2a_2 r + a_1 \\
 p_2(r) &= -\text{rem}(p_0/p_1) = c_6 r^6 + c_5 r^5 + c_4 r^4 + c_3 r^3 + c_2 r^2 + c_1 r + c_0 \\
 &\quad \text{(the negative of the remainder of division of } p_0 \text{ by } p_1) \\
 p_3(r) &= -\text{rem}(p_1/p_2) = d_5 r^5 + d_4 r^4 + d_3 r^3 + d_2 r^2 + d_1 r + d_0 \\
 p_4(r) &= -\text{rem}(p_2/p_3) = e_4 r^4 + e_3 r^3 + e_2 r^2 + e_1 r + e_0 \\
 p_5(r) &= -\text{rem}(p_3/p_4) = f_3 r^3 + f_2 r^2 + f_1 r + f_0 \\
 p_6(r) &= -\text{rem}(p_4/p_5) = g_2 r^2 + g_1 r + g_0 \\
 p_7(r) &= -\text{rem}(p_5/p_6) = h_1 r + h_0 \\
 p_8(r) &= -\text{rem}(p_6/p_7) = i_0
 \end{aligned}$$

Where the explicit expression of the corresponding co-efficients a_i, c_i, d_i, \dots has been provided in the equation (11) and in the Appendix - II. If one needs to figure out the number of roots of p_0 in $[a, b]$, the number of sign changes in the sequence $p_0(a), p_1(a), p_2(a), p_3(a), p_4(a), p_5(a), p_6(a), p_7(a), p_8(a)$ is to be counted and let us call it $v(p_0, a)$. Similarly, the count the number of sign changes in the sequence $p_0(b), p_1(b), p_2(b), p_3(b), p_4(b), p_5(b), p_6(b), p_7(b), p_8(b)$ is to be called as $v(p_0, b)$. Then, the number of roots of p_0 in $[a, b]$ is $v(p_0, a) - v(p_0, b)$.

It is important to note that direct application of the Sturm's theorem may not always be sufficient since some of the roots may yield a negative energy for \mathcal{E} (since the \mathcal{E} equation was squared to get the polynomial). Since we are interested in accretion with the positive positive Bernoulli's constant, to get positive values of the energy, we must impose the condition that

$$\gamma - (1 + c_s^2) \geq 0, \quad (22)$$

which is the term present in \mathcal{E} which could go negative. This introduces the condition that $\frac{p(r)}{q(r)} \geq 0$, where $p(r)$ and $q(r)$ are 4th order polynomials given by,

$$p(r) = 6(\gamma - 1)r^4 - (11\gamma - 13)r^3 - (5\gamma - 3)\lambda^2 r^2 + 2(9\gamma - 5)\lambda^2 r - 8(2\gamma - 1)\lambda^2, \quad (23a)$$

$$q(r) = 6r^4 - 12r^3 - 4\lambda^2 r^2 + 14\lambda^2 r - 12\lambda^2. \quad (23b)$$

To find the region where this happens, one has to find the 4 roots of each of $p(r)$ and $q(r)$ – which is analytically possible since roots of quartics are analytically solvable. Once the roots are obtained it is a trivial matter to check for what regions the rational function is positive.

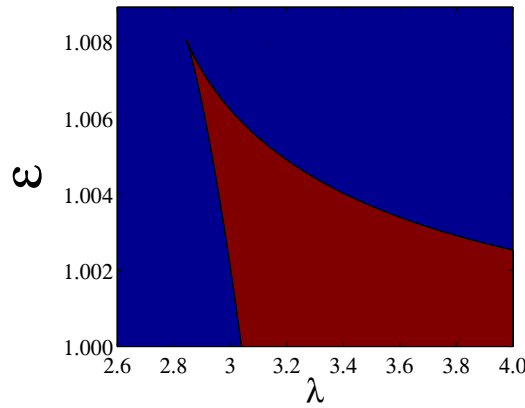


Fig. 1 The lighter region (online version red) corresponds to 3 roots and the darker shade (online version blue) indicates 1 root only. The value of γ is $4/3$.

A simplified version for the above mentioned procedure to find the positivity condition is as follows:

We would like to find out the intervals in which $p(r)/q(r) > 0$ where $p(r)$ and $q(r)$ are quartic polynomials. We factorize $p(r) = (r - r_1)(r - r_2)(r - r_3)(r - r_4)$ and $q(r) = (r - s_1)(r - s_2)(r - s_3)(r - s_4)$ using the algorithm for finding roots of a quartic. If the roots are all real, we note down the sign changes of each factor to the right and left of each root and find out the intervals where the rational function is positive. If there are complex roots, they come in complex conjugates, since the coefficients of the polynomials are real. Say, if r_3 is complex and r_4 is its complex conjugate, then the part $(r - r_3)(r - r_4) = r^2 - (r_3 + r_4)r + r_3 r_4$ does not change sign since it is non-zero on the real line. It is easy to determine its sign.

To demonstrate the procedure described above, the number of roots of the 8th order polynomial p_0 (in the Sturm sequence) within the admissible range of \mathcal{E} , λ and γ (usually by keeping the value of γ to be fixed to obtain a two dimensional parameter space) are evaluated explicitly and that shows two distinct regions in $\mathcal{E} - \lambda$ space (see

Fig. 1). The wedge shaped region corresponds to 3 roots implying 3 critical points and the rest of the parametric space corresponds to single root implying only one critical point. This feature emerging from the above mentioned algorithm, exactly conforms with the numerical results (using the explicit root finding methods) available in the current literature [32]. It may be worthwhile to mention here that in addition to these roots there exists another root for the the whole range of parameter space shown in the Fig. 1 that is located very near to the event horizon (i.e. within 1–1.5 times Schwarzschild radius), but being a centre it is physically untenable to be a sonic point (a critical point through which a physical accretion solution, connecting the event horizon with to infinity, can pass) and hence has always been justifiably ignored in the literature.

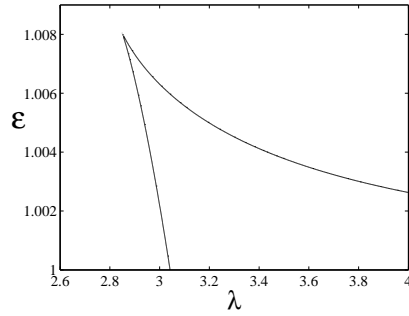


Fig. 2 Boundary of transition: Contour line $\det(S) = 0$ (for $\gamma = 4/3$).

The transition boundaries from n_1 number of roots to n_2 number of roots, in the parameter space, can be more easily obtained using catastrophe theory. The boundaries of the region in the parameter space permitting transition of number of critical points in this case are associated with saddle-centre bifurcation or merging of a pair of roots of the equation (Eq.11). Now all these equations are polynomial equations. As a general rule the discriminant of a polynomial,

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad (24)$$

can be expressed as in terms of its roots, x_i 's, as

$$D = a_n^{n-2} \prod_{i < j} (x_i - x_j)^2. \quad (25)$$

The discriminant may be expressed as the determinant of a matrix called the Sylvester matrix (see, e.g., <http://mathworld.wolfram.com/PolynomialDiscriminant.html>, and

references therein),

$$S = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0 \dots & \dots & 0 \\ 0 & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0 \dots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \dots & 0 & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & 1a_1 & 0 & \dots & \dots & 0 \\ 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & 1a_1 & 0 & \dots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & \dots & 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & 1a_1 \end{bmatrix}, \quad (26)$$

up to a factor.

Putting $n = 8$, $\det(S)$ will be zero on the above mentioned boundaries and actually it is so. Here the plot of $\det(S) = 0$ for the polytropic flow (i.e. for the polynomial in r_c in Eq.11) in \mathcal{E} - λ space is shown in Fig.2. The curve exactly conforms with the corresponding boundary curve in Fig.1, drawn on the basis of the previous method. So this procedure may be thought of as a much easier alternative to find the multi-critical parametric values; though this method cannot give the exact number of critical points in each region of the parameter space.

5 Discussion

Our methodology is based on the algebraic form of the first integral obtained by solving the radial momentum equation (the Euler equation to be more specific, since we are confined to the inviscid flow only). The structure for such a first integral has to be a formal polynomial with appropriate constant co-efficients. For general relativistic accretion in the Kerr metric, the expression for the energy first integral can not be reduced to such a polynomial form (see, e.g., [33] for the detail form of such algebraic expression). Hence, the Sturm's generalized chain can not be constructed for such accretion flow. Alternative methodology are required to investigate the multi-critical behaviour for such kind of accretion.

Using the method illustrated in this work, it is possible to find out how many critical points a transonic black hole accretion flow can have. It is thus possible to predict whether such accretion flow can have multi-critical properties for a certain specific value/domain of the initial boundary conditions. It is, however, not possible to investigate, using the eigenvalue analysis as illustrated in [8,12], the nature of such critical points - i.e., whether they are of saddle type or are of centre type, since such prediction requires the exact location of the critical points (the value of the roots of the polynomial). However, the theory of dynamical systems ensures that no two consecutive critical points be of same nature (both saddle or both centre). On the other hand, our experience predicts (it is rather a documented fact) that for all kind of multi-critical black hole accretion, irrespective of the equation of state, the space time geometry or the flow configuration used, one has two saddle type critical points and one centre type critical point flanked by them (see, e.g., [33] and [34] for

further detail). Hence if the application of the Sturm's generalized chain ensures the presence of three critical points, we can say that out of those three critical points, accretion flow will have two saddle type critical points, hence a specific subset of the solution having three roots corresponding to the first integral polynomial, can make transonic transition for more than one times, if appropriate conditions for connecting the flow through the outer critical point and for flow through the inner critical points are available, see, e.g., [33] for further discussion.

In this work we have considered only inviscid accretion. Our methodology of investigating the multi-critical properties, however, is expected to be equally valid for the viscous accretion disc as well. For the viscous flow, the radial momentum conservation equation involving the first order space derivative of the dynamical flow velocity will certainly provide a first integral of motion upon integration. Because of the fact that a viscous accretion disc is not a non-dissipative system, such constant of motion, however, can never be identified with the specific energy of the flow. The integral solution of the radial momentum equation would then be an algebraic expression of various flow variables and would perhaps involve certain initial boundary conditions as well. Such an algebraic expression would actually be a constant of motion. What exactly would that expression physically signify, would definitely be hard to realize. However, one may perhaps arbitrarily parameterize that conserved algebraic expression using some astrophysically relevant outer boundary conditions, and if such algebraic expressions can finally be reduced, using the appropriate critical point conditions, to an algebraic polynomial form of the critical points, construction of a generalized Sturm chain can be made possible to find out how many critical points such an accretion flow can have subjected to the specific initial boundary condition. Since for accretion onto astrophysical black holes, having multiple critical points is a necessary (but not sufficient) condition to undergo shock transition, one can thus analytically predict, at least to some extent, which particular class of viscous accretion disc are susceptible for shock formation phenomena.

Our work, as we believe, can have a broader perspective as well, in the field of the study of dynamical systems in general. For a first order autonomous dynamical system, provided one can evaluate the critical point conditions, the corresponding generalized n th degree algebraic equation involving the position co-ordinate and one (or more) first integral of motion can be constructed. If such algebraic equation can finally be reduced to a n th degree polynomial with well defined domain for the constant co efficient, one can easily find out the maximal number of fixed points of such dynamical systems.

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6 Appendix - I : Proof of the Sylvester's theorem:

First note that the Sturm sequence (f_0, \dots, f_k) is (up to signs) equal to the sequence obtained from the Euclidean algorithm. Define a new sequence (g_0, \dots, g_k) by $g_i = f_i/f_k$ for $i \in \{0, \dots, k\}$. Note that the number of sign changes in $(f_0(x), f_1(x))$ (resp. $(f_{i-1}(x), f_i(x), f_{i+1}(x))$) and the number of sign changes in $(g_0(x), g_1(x))$ (resp. $(g_{i-1}(x), g_i(x), g_{i+1}(x))$) coincide for any x which is not a root of f . Note also that the roots of g_0 are exactly the roots of f which are not roots of g . Observe that for $i \in 0, \dots, k, g_{i-1}$ and g_i are relatively prime. We consider, now, how $v(f, g; x)$ behaves when x passes through a root c of a polynomial g_i . If c is a root of g_0 , then it is not a root of g_1 . We write $f'(c_-) > 0$ (resp. < 0) if f' is positive (resp. negative) immediately to the left of c . The sign of $f'(c_+)$ is defined similarly. Now we recall the following result: if R is a real closed field, $f \in R[X]$, $a, b \in R$ with $a < b$ and if the derivative f' is positive (resp. negative) on $]a, b[$, then f is strictly increasing (resp. strictly decreasing) on $[a, b]$. Then, according to the signs of $g(c)$, $f'(c_-)$ and $f'(c_+)$ we have the following 8 cases:

$$g(c) > 0, f'(c_-) > 0, f'(c_+) > 0$$

	c_-	c	c_+
f	-	0	+
$f'g$	+		+

$$g(c) < 0, f'(c_-) > 0, f'(c_+) > 0$$

	c_-	c	c_+
f	-	0	+
$f'g$	-		-

$$g(c) > 0, f'(c_-) < 0, f'(c_+) > 0$$

	c_-	c	c_+
f	+	0	+
$f'g$	-		+

$$g(c) < 0, f'(c_-) < 0, f'(c_+) > 0$$

	c_-	c	c_+
f	+	0	+
$f'g$	+		-

$$g(c) > 0, f'(c_-) > 0, f'(c_+) < 0$$

	c_-	c	c_+
f	$-$	0	$-$
$f'g$	$+$		$-$

$$g(c) < 0, f'(c_-) > 0, f'(c_+) < 0$$

	c_-	c	c_+
f	$-$	0	$-$
$f'g$	$-$		$+$

$$g(c) > 0, f'(c_-) < 0, f'(c_+) < 0$$

	c_-	c	c_+
f	$+$	0	$-$
$f'g$	$-$		$-$

$$g(c) < 0, f'(c_-) < 0, f'(c_+) < 0$$

	c_-	c	c_+
f	$+$	0	$-$
$f'g$	$+$		$+$

In every as x passes through c , the number of sign changes in $(f_0(x), f_1(x))$ decreases by 1 if $g(c) > 0$, and increases by 1 if $g(c) < 0$. If c is a root of g_i with $i = 1, \dots, k$, then it is neither a root of g_{i-1} nor a root of g_{i+1} , and $g_{i-1}(c)g_{i+1}(c) < 0$, by the definition of the sequence. Passing through c does not lead to any modification of the number of sign changes in $(f_{i-1}(x), f_i(x), f_{i+1}(x))$ in this case.

Proof of the Sturm's theorem: Using $g = 1$ in previous theorem.

7 Appendix - II: Explicit expressions for the co-efficients for the Sturm chain constructed for the relativistic axisymmetric accretion

$$\begin{aligned}
c_6 &= \frac{7a_7^2}{64a_8} - a_6/4, \\
c_5 &= \frac{3a_6a_7}{32a_8} - \frac{3}{8}a_5 \\
c_4 &= \frac{5a_5a_7}{64a_8} - a_4/2 \\
c_3 &= \frac{2a_4a_7}{32a_8} - \frac{5}{8}a_3 \\
c_2 &= \frac{3a_3a_7}{64a_8} - \frac{3}{4}a_2 \\
c_1 &= \frac{a_2a_7}{32a_8} - \frac{7}{8}a_1 \\
c_0 &= \frac{a_1a_7}{64a_8} - a_0
\end{aligned}$$

$$\begin{aligned}
d_5 &= \frac{8a_8c_4}{c_6} + \left(\frac{7a_7}{c_6} - \frac{8c_5a_8}{c_6^2}\right)c_5 - 6a_6 \\
d_4 &= \frac{8a_8c_3}{c_6} + \left(\frac{7a_7}{c_6} - \frac{8c_5a_8}{c_6^2}\right)c_4 - 5a_5 \\
d_3 &= \frac{8a_8c_2}{c_6} + \left(\frac{7a_7}{c_6} - \frac{8c_5a_8}{c_6^2}\right)c_3 - 4a_4 \\
d_2 &= \frac{8a_8c_1}{c_6} + \left(\frac{7a_7}{c_6} - \frac{8c_5a_8}{c_6^2}\right)c_2 - 3a_3 \\
d_1 &= \frac{8a_8c_0}{c_6} + \left(\frac{7a_7}{c_6} - \frac{8c_5a_8}{c_6^2}\right)c_1 - 2a_2 \\
d_0 &= \left(\frac{7a_7}{c_6} - \frac{8c_5a_8}{c_6^2}\right)c_0 - a_1
\end{aligned}$$

$$\begin{aligned}
e_4 &= \frac{d_3c_6}{d_5} + \left(\frac{c_5}{d_5} - \frac{d_4c_6}{d_5^2}\right)d_4 - c_4 \\
e_3 &= \frac{d_2c_6}{d_5} + \left(\frac{c_5}{d_5} - \frac{d_4c_6}{d_5^2}\right)d_3 - c_3 \\
e_2 &= \frac{d_1c_6}{d_5} + \left(\frac{c_5}{d_5} - \frac{d_4c_6}{d_5^2}\right)d_2 - c_2 \\
e_1 &= \frac{d_0c_6}{d_5} + \left(\frac{c_5}{d_5} - \frac{d_4c_6}{d_5^2}\right)d_1 - c_1 \\
e_0 &= \left(\frac{c_5}{d_5} - \frac{d_4c_6}{d_5^2}\right)d_0 - c_0
\end{aligned}$$

$$\begin{aligned}
f_3 &= \frac{e_2 d_5}{e_4} + \left(\frac{d_4}{e_4} - \frac{e_3 d_5}{e_4^2} \right) e_3 - d_3 \\
f_2 &= \frac{e_1 d_5}{e_4} + \left(\frac{d_4}{e_4} - \frac{e_3 d_5}{e_4^2} \right) e_2 - d_2 \\
f_1 &= \frac{e_0 d_5}{e_4} + \left(\frac{d_4}{e_4} - \frac{e_3 d_5}{e_4^2} \right) e_1 - d_1 \\
f_0 &= \left(\frac{d_4}{e_4} - \frac{e_3 d_5}{e_4^2} \right) e_0 - d_0.
\end{aligned}$$

$$\begin{aligned}
g_2 &= \frac{f_1 e_4}{f_3} + \left(\frac{e_3}{f_3} - \frac{f_2 e_4}{f_3^2} \right) f_2 - e_2 \\
g_1 &= \frac{f_0 e_4}{f_3} + \left(\frac{e_3}{f_3} - \frac{f_2 e_4}{f_3^2} \right) f_1 - e_1 \\
g_0 &= \left(\frac{e_3}{f_3} - \frac{f_2 e_4}{f_3^2} \right) f_0 - e_0
\end{aligned}$$

$$\begin{aligned}
h_1 &= \frac{g_0 f_3}{g_2} + \left(\frac{f_2}{g_2} - \frac{g_1 f_3}{g_2^2} \right) g_1 - f_1 \\
h_0 &= \left(\frac{f_2}{g_2} - \frac{g_1 f_3}{g_2^2} \right) g_0 - f_0
\end{aligned}$$

$$i_0 = \left(\frac{g_1}{h_1} - \frac{h_0 g_2}{h_1^2} \right) h_0 - g_0$$

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